

Bose condensation in two dimensions with disorder: Gross-Pitaevskii approach

Dmitri E. Nikonov and Ata Imamoğlu

*Department of Electrical and Computer Engineering
and Center for Quantized Electronic Structures (QUEST),
University of California, Santa Barbara, California 93106-9560*

February 1, 2008

Abstract

Bose condensation of interacting bosons in a two-dimensional random potential is studied. The Gross-Pitaevskii equation is solved to determine the spatially-varying order parameter and the localization length as a function of the disorder, the interaction strength, and the condensate density. A finite temperature of condensation is obtained thereby. The results are applied to determination of the superradiant decay of excitons in a GaAs quantum well.

PACS number(s): 03.75.Fi, 71.23.An, 75.40.Cx

Bose condensation in a disordered medium was the topic of extensive research starting a decade ago [1, 2]. It was mainly concerned with the superfluidity of liquid ^4He in porous media or on substrates [3, 4, 5] or with the superconductor-insulator transition in high critical temperature superconductors [6, 7, 8, 9]. In most cases, the Hubbard Hamiltonian of interacting bosons on a lattice is employed. This model is treated via a mean-field theory [10], via the Bogolyubov approximation [11], in the approximation of Gaussian fluctuations [12], by the Monte Carlo numerical methods [13, 14, 15], or by renormalization group methods [16, 17].

Bose condensation in other systems is often treated via the Gross-Pitaevskii equation (nonlinear Schrödinger equation) for the condensate [18]. The advantages of this approach are that one immediately obtains the order parameter of a non-uniform condensate and that the non-physical features related to the integer occupancy number of a lattice site do not appear. The connection between two approaches was made in Refs. [4, 11].

Our interest in Bose condensation in a two-dimensional space is primarily motivated by the ongoing experiments aimed at observing quantum degeneracy of excitons in quantum wells [19]. Since in two or smaller dimensions the critical temperature of condensation would be zero in a uniform infinite space, only localization by geometrical size, or, in our case, localization by disorder can cause condensation at a finite temperature. This phase, consisting of a number of separate and presumably phase-uncorrelated “lakes” of Bose condensate is more properly termed “Bose glass” [3].

For excitons, the order parameter is proportional to the polarization of the medium. Therefore Bose condensation has a profound influence on the optical properties of excitons, especially on the superradiant decay. The smallest of the exciton mean free path and the localization length gives the coherence area of an exciton. It was shown in Ref. [20, 21, 22] that the coherence area is related to the linewidth and the radiative lifetime of excitons. Bjork et al. discussed it in terms of superradiance of excitons [23]. A sharp increase of the exciton oscillator strength was suggested as a criterion of Bose condensation of excitons [19, 24]. The oscillator strength reaches saturation as the coherence length approaches the wavelength. Other approaches (Gerrit Bauer, private communication) predict a divergence of the coherence area as the temperature decreases down to some critical temperature, and a constant coherence area below this temperature. In view of this controversy, it is important to give an intuitive treatment of the oscillator strength and the superradiant decay.

In this paper we numerically solve the Gross-Pitaevskii equation for the order parameter of the condensate in a disordered two-dimensional medium modeled by a set of potential wells with their depths depending on the widths. [The authors understand the limited nature of such an assumption, but stress that such a potential might be applicable to quantum dots defined by the variation of a quantum-well width]. In this way we determine the localization size as a function of the disorder strength, the interaction strength, and the condensate density. This relation along with the integral of the Bose distribution for a localized condensate allow us to obtain the temperature of condensation for a given density of bosons. The localization length determines the oscillator strength and the increase of the rate of superradiant decay.

We describe the condensate of interacting bosons via the stationary Gross-Pitaevskii equation for the long-range order parameter ψ [18]

$$-\frac{\hbar^2}{2M}\nabla^2\psi + V(\mathbf{r})\psi + U\psi|\psi|^2 = \mu\psi, \quad (1)$$

where M is the mass of a boson, U is the interaction strength, μ is the chemical potential, and V is the external potential. This model implies that we take the lifetime of the boson particles to be infinite and neglect scattering.

This would be a good approximation for indirect excitons (having the electron and the hole in different quantum wells) in a direct-bandgap semiconductors. Equation (1) also implies that the exciton-exciton interaction can be modeled by a short-range s -wave scattering. Even though this assumption is at best questionable, we make it for the sake of simplicity.

To incorporate a random potential, caused, e.g., by impurities or fluctuations of the quantum well thickness, we adopt the following model. The whole two-dimensional space is divided into a set of “lakes” of condensate of the characteristic size L_c . The condensate in each of them is presumed to be trapped by a cylindrical potential well of a radius L to be determined: following Ref. [25] we choose the depth of this well to be inversely proportional to its radius

$$V(\mathbf{r}) = -\frac{\xi}{L}\theta(L - |\mathbf{r}|), \quad (2)$$

where ξ is the parameter characterizing the strength of the disorder. The average density of bosons in the condensate is n_c . The average number of bosons in each lake

is $N_c = n_c L_c^2$. We impose a condition to normalize the number of bosons in each condensate lake

$$\int |\psi|^2 d\mathbf{r} = n_c L_c^2, \quad (3)$$

and we identify L_c with the coherence length defined by the following relation

$$\int |\psi|^4 d\mathbf{r} = n_c^2 L_c^2. \quad (4)$$

The integrals have infinite limits.

In order to reveal the self-similarity of the solutions, let us introduce a constant length scale L_0 and correspondingly express all densities in terms of $1/L_0^2$. On the other hand, the coordinates will be expressed in terms of the well width L and the wavefunction will be expressed in terms of $1/L$. Then the Gross-Pitaevskii equation in dimensionless variables becomes

$$-\nabla^2 \psi - \xi_0 L \theta(1 - |\mathbf{r}|) \psi + u \psi |\psi|^2 = \mu_0 L^2 \psi, \quad (5)$$

where the dimensionless parameters

$$\xi_0 = \frac{2M\xi L_0}{\hbar^2}, \quad (6)$$

$$u = \frac{2MU}{\hbar^2}, \quad (7)$$

$$\mu_0 = \frac{2M\mu L_0^2}{\hbar^2}. \quad (8)$$

The parameter l_h is called “healing length” of the condensate,

$$l_h^2 = \frac{2\hbar^2}{MU n_c} = \frac{4}{u n_c}. \quad (9)$$

By comparing the first and the third terms in (5), one can estimate that the order parameter can appreciably change only over the dimensionless length of the order of l_h . If u is large, the healing length is less than the interparticle separation, and if u is small, it is larger than the interparticle separation. Under this change of variables (3) and (4) become

$$\int |\psi|^2 d\mathbf{r} = n_c L_c^2, \quad (10)$$

$$\int |\psi|^4 d\mathbf{r} = n_c^2 L_c^2 L^2. \quad (11)$$

At any given strength of disorder ξ , for convenience, we chose the length scale

$$L_0 = \frac{\hbar^2}{2M\xi} \quad (12)$$

so that $\xi_0 = 1$. Besides, we notice a self-similarity of the set of the equations (5)-(11) under the transformation with an arbitrary positive a

$$u \rightarrow u' = u/a, \quad n \rightarrow n' = an, \quad \psi \rightarrow \psi' = \sqrt{a}\psi. \quad (13)$$

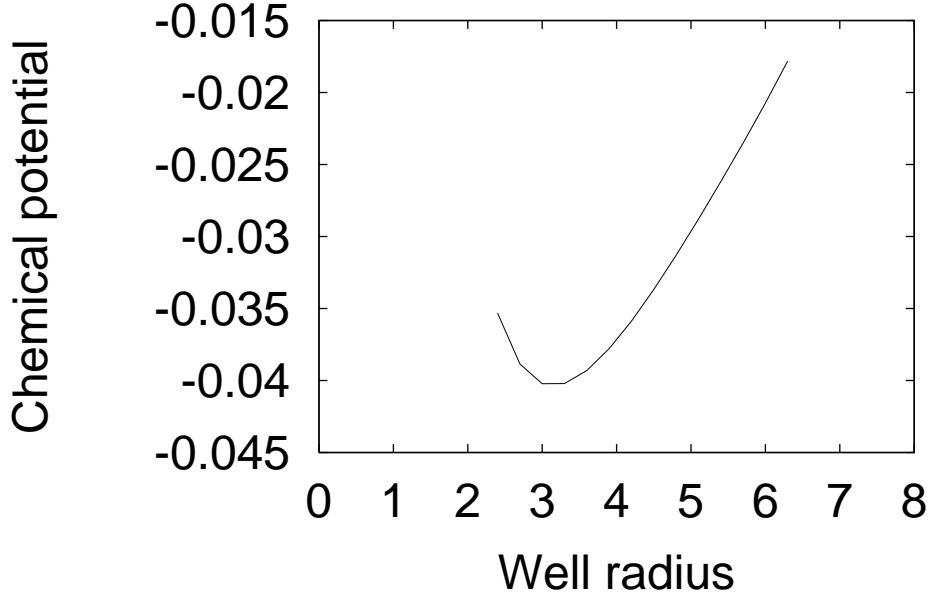


Figure 1: Dependence of the chemical momentum of the condensate on the trapping well width at $u = 1$, $n = 0.06$.

Therefore we numerically solve the set of equations for $u = 1$ and obtain the general solution by rescaling according to (13).

The solution of the dimensionless equations (5) - (11) determines the dependence of the chemical potential of the condensate on the trapping radius L . An example of it for $u = 1$ is shown in Fig. 1. For the trapping radii outside the region on the plot no localized solutions (with $\mu < 0$) were found. At $u = 0$, a linear Schrödinger equation, solutions exist for any radius L of the well. As the density of bosons, or, equivalently, the nonlinear coefficient u , increases, the interaction of bosons screens the random potential more strongly [4]. As a result, the range of L , where localized solutions are possible, narrows until it totally disappears. The radius L is determined from the solution of (1) which minimizes the chemical potential μ . Note that extended condensate solutions are not a limit of localized solutions. The former have a positive chemical potential $\mu = un_c$. The latter must have negative μ to ensure decrease $\psi \rightarrow 0$ at $|x| \rightarrow \infty$.

Based on these solutions, we seek the dependence of the condensate size L_c on the average condensate density n_c . The numerical result for $u = 1$ is shown in Fig. 2. With a good accuracy at all densities this dependence can be approximated by

$$L_c = \alpha(n_g - n)^\beta \quad (14)$$

with $\alpha = 5.4$, $\beta = -0.1317$, and $n_g = 0.074$. Thus below the critical density (proportional to the dimensionless constant n_g), the condensate exists as a Bose glass with localized “lakes” of condensate, and, above this density, an extended Bose condensate is possible at zero temperature. This universal dependence will be used later while treating bosons with different nonlinear coefficient u and at non-zero temperatures.

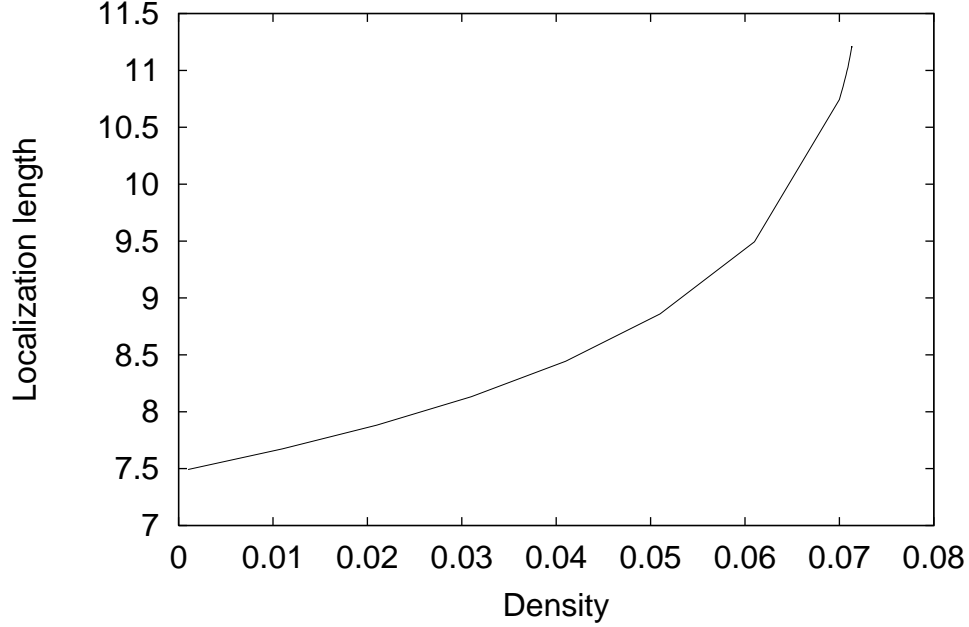


Figure 2: Dependence of the localization length of the condensate on its density at $u = 1$.

Now we apply our model to the case of excitons in GaAs quantum wells. Let us estimate the interaction strength U for excitons in the ideal two-dimensional case. It is proportional to the exciton-exciton interaction term, i.e., the Coulomb exchange integral in the exciton ground (1s) state, which, according to [26] is equal to

$$I = \sum_{kk'} [\phi_{1s}(k') - \phi_{1s}(k)] \phi_{1s}^*(k) |\phi_{1s}(k)|^2 V(k - k') = 4\pi a_0^2 (1 - \frac{315\pi^2}{4096}) E_0, \quad (15)$$

where $\phi_{1s}(k)$ is the wavefunction of the 1s exciton state and $V(k)$ is the Coulomb potential in the momentum representation, a_0 is the Bohr radius and E_0 is the binding energy of an exciton in two dimensions given in [26]. We have neglected screening of the Coulomb potential by other excitons, which is possible to do for small densities (interparticle separation much larger than a_0). At higher densities (15) overestimates the strength of interaction u .

The estimate of I for indirect excitons will be inserted here.

Therefore we obtain that

$$u = \frac{4MI}{\hbar^2} \approx \frac{6.06M}{m_r}, \quad (16)$$

bearing in mind that $M = m_e + m_h$ and $m_r = m_e m_h / M$. For the case of GaAs, where $m_e = 0.0665m_0$, $m_h = 0.377m_0$ and m_0 is the mass of a free electron, we obtain $u \approx 47$, which means strong boson-boson interaction. It is the opposite of Bose condensation in atomic traps in which one encounters a weak interaction,

$$u \sim \frac{a_S}{L_{\text{trap}}}, \quad (17)$$

where a_S is the scattering length in s -wave collisions, and L_{trap} is the size of the trap.

It is well-known that the critical temperature of Bose condensation in two dimensions is zero for an infinite space. Therefore only a localized (Bose glass) rather than an extended condensate can exist at non-zero temperatures. Only a fraction (if any) of bosons will be in the condensate and be described by the Gross-Pitaevskii equation.

¹ We need to add the relations describing the statistics of bosons outside the condensate. Following [28] we obtain for a two-dimensional localized Bose gas with the average density n

$$1 - \frac{n_c}{n} = \frac{\Lambda_{cr}^2}{\Lambda^2}, \quad (18)$$

$$n\Lambda_{cr}^2 = \log\left(\frac{2L_c^2}{\Lambda_{cr}^2}\right), \quad (19)$$

where k_B is the Boltzmann constant, the de Broglie wavelength corresponding to the temperature T is

$$\Lambda = \sqrt{\frac{2\pi\hbar^2}{Mk_BT}}, \quad (20)$$

and Λ_{cr} is the wavelength corresponding to a critical temperature for the given localization size L_c . Here we again assume the same scaling of lengths and densities. Equations (18) and (19) together with the rescaled (14)

$$L_c = \alpha(n_g - un_c)^\beta \quad (21)$$

allow us to determine the condensate density n_c and the localization length L_c for the given concentration and temperature. We see that the condensate density cannot exceed the critical value n_g/u .

The correlation length allows us to obtain the collective dipole of the condensate and the rate of radiative decay. Exciton condensate is similar to a collective of excitons excited by a laser. The differences are: 1) that the direction and the phase of the dipole are not given by the laser, but undetermined until a measurement is made on the condensate; 2) the order parameter does not contain a spatial dependence associated with the wavevector of the exciting laser. With these modifications, we apply the results of Bjork et al. [23]. The number of modes available for excitons are

$$N_e = \frac{8a_c^2}{a_0^2}, \quad (22)$$

where the condensate radius is such that $\pi a_c^2 = L_c^2$. In the case of a thin quantum well and large N_e , according to [29], the power of emission per unit angle is proportional to $I(\phi, \chi)\Gamma(\phi)$, where the usual dipole pattern

$$I((\phi, \chi) = \cos^2 \chi + \sin^2 \chi \cos^2 \phi, \quad (23)$$

and the factor of the collective emission is

$$\Gamma(\phi) = \left(\frac{2J_1(ka_c \sin \phi)}{ka_c \sin \phi}\right)^2 \quad (24)$$

¹We disregard the possibility of the Kosterlitz-Thouless transition, see e.g. [27].

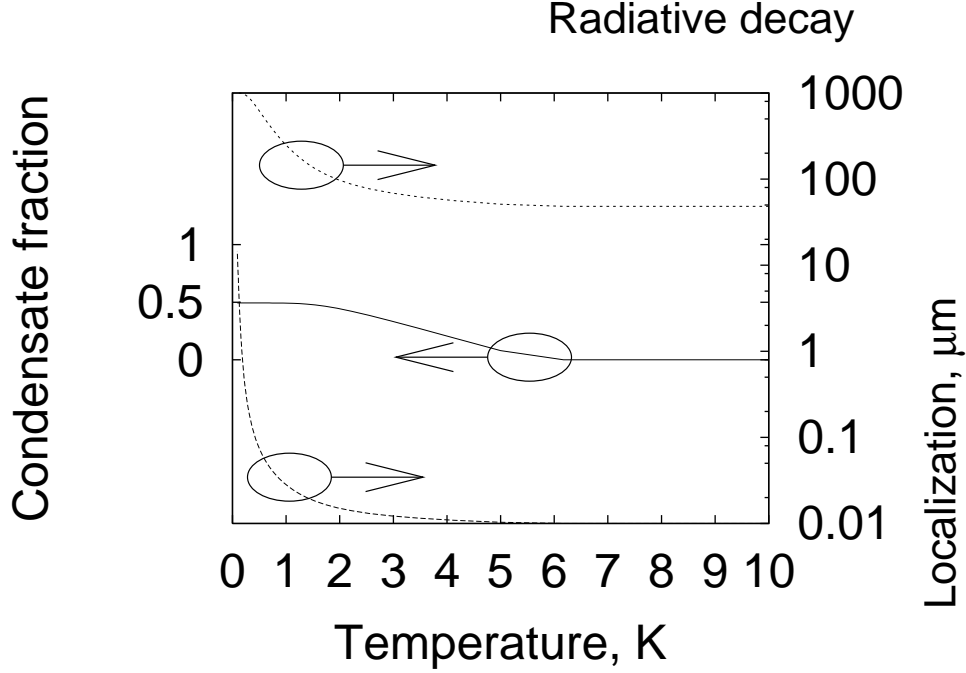


Figure 3: Fraction of the bosons in the condensate (middle curve) the localization length in μm (lower curve) and the factor of enhancement of radiative decay rate (upper curve) as functions of temperature for $u = 47$ and $L_0 = 10^{-8}\text{m}$: a) $n = 1.2 \times 10^{10}\text{cm}^{-2}$.

Here $k = 2\pi/\lambda$, and λ is the wavelength in the material (228nm in GaAs), ϕ is the angle between the direction of emission and the orthogonal to the well. The cooperativity parameter, which determines how well the radiators are localized, is

$$\mu_c = \frac{3}{8k^2a_c^2} \int_0^\pi \sin \phi d\phi (1 + \cos^2 \phi) \Gamma(\phi). \quad (25)$$

For small condensates ($ka_c \ll 1$) it tends to 1, and for large condensates it tends to $3/(ka_c)^2$. The rate of superradiant emission of excitons γ is increased compared to the bulk rate of free-electron-hole recombination γ_0 by the enhancement factor

$$\gamma = \mu_c N_e \gamma_0. \quad (26)$$

For small condensates the emission rate grows as a_c^2 to the same extent in all directions. As the size a_c of the condensate grows, the enhancement factor tends to a constant. However the angular distribution becomes strongly peaked at $\phi = 0$ where it still grows as a_c^2 .

The results of the numerical solution of (18)-(21) are shown in Fig. 3-5. For illustrative purposes we restore the dimension of length by taking a specific disorder strength which corresponds to $L_0 = 10^{-8}\text{m}$. We see that the condensate (a non-vanishing share of the condensate density) appears below a well-defined temperature. If the total density of bosons is below the critical density, the fraction of the condensate tends to 1 as the temperature decreases, and the localization length tends to a small finite value,

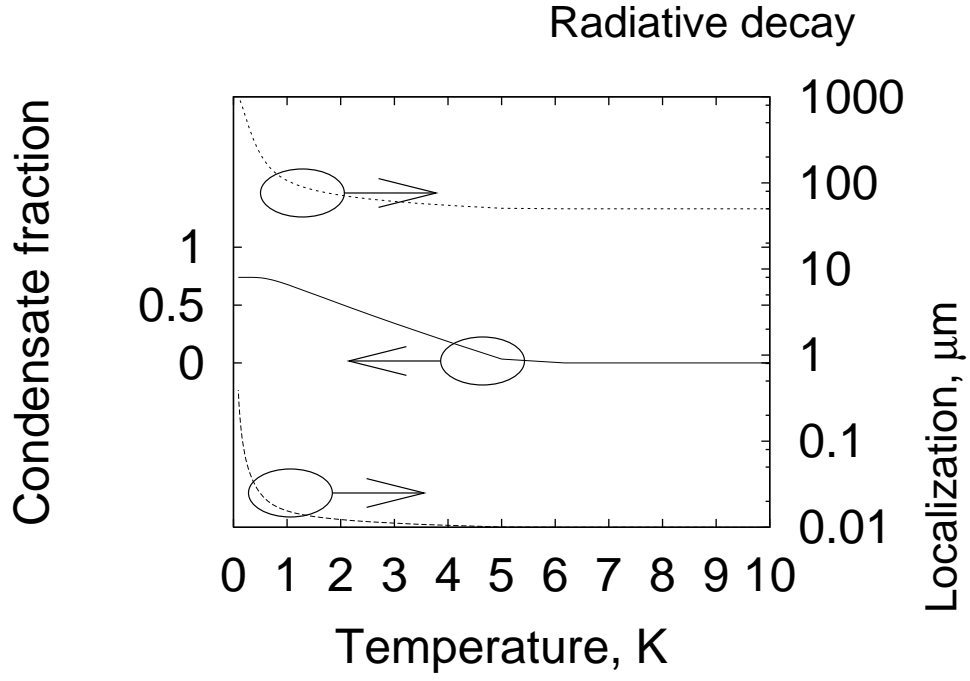


Figure 4: Same as above: b) $n = 0.8 \times 10^{10} \text{cm}^{-2}$.

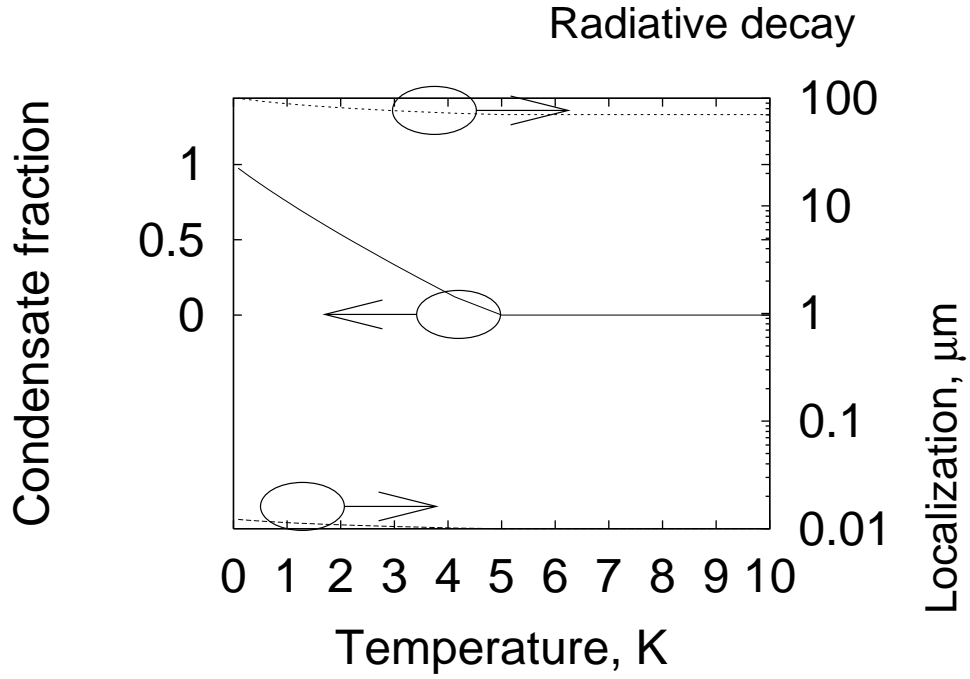


Figure 5: Same as above: c) $n = 0.4 \times 10^{10} \text{cm}^{-2}$.

see Fig. 5. If the total density is above the critical one, the density of the condensate tends to that critical value so that the condensate fraction is always less than 1. The localization length tends to infinity as the temperature decreases and its value strongly depends on the total density, see Fig. 3-3. However this increase does not become noticeable until temperatures several times below the condensation temperature. The enhancement factor grows appreciably until it reaches the limiting value as per reasoning above.

In conclusion, we have presented a theoretical description of Bose condensation in a random potential in two dimensions. The disorder is believed to be sufficiently strong so that the Kosterlitz-Thouless transition the effects of the sample size can be disregarded. We argued then that only localized condensates (Bose glass) can exist at non-zero temperatures. A macroscopic occupation of the condensate appears in a sharp transition below some condensation temperature. The average density in the condensate always stays below a certain critical density determined by the disorder strength. As the temperature decreases, the localization length, and consequently the coherence area, grow indefinitely if the average density of bosons is higher than the critical density. The increase of the radiative decay rate reaches its limit when the correlation length reaches the photon wavelength in the medium. The increase in the rate of superradiant decay and, especially, in the power of emission orthogonal to the well, is a reliable indicator of Bose condensation.

This work is supported by NSF, Packard Foundation, and the Center for Quantized Electronic Structures (QUEST), an NSF Science and Technology Center. We thank Leonid Butov for enlightening discussions and making available his experimental results before publication.

References

- [1] M. Ma, B. I. Halperin, and P. A. Lee, Phys. Rev. B **34**, 3136 (1986).
- [2] D. S. Fisher and M. P. A. Fisher, Phys. Rev. Lett. **61**, 1847 (1988).
- [3] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
- [4] D. K. K. Lee and J. M. F. Gunn, J. Phys. **2**, 7753 (1990).
- [5] W. Krauth, N. Trivedi, and D. Ceperley, Phys. Rev. Lett. **67**, 2307 (1991).
- [6] M. Ma and P. A. Lee, Phys. Rev. B **32**, 5658 (1985).
- [7] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1990).
- [8] A. Gold, Physica C **190**, 483 (1992).
- [9] D. R. Nelson and V. M. Vinokur, Phys. Rev. Lett. **68**, 2398 (1992).
- [10] K. Sheshadri, H. R. Krishnamurthy, R. Pandit, and T. V. Ramakrishnan, Europhys. Lett. **22**, 257 (1993).
- [11] K. G. Singh and D. S. Rokhsar, Phys. Rev. B **49**, 9013 (1994).

- [12] L. Z. P. Nisamaneephong and M. Ma, Phys. Rev. Lett. **71**, 3830 (1993).
- [13] R. T. Scalettar, G. G. Batrouni, and G. T. Zimanyi, Phys. Rev. Lett. **66**, 3144 (1991).
- [14] M. Makivicć, N. Trivedi, and S. Ullah, Phys. Rev. Lett. **71**, 2307 (1993).
- [15] T. Onogi and Y. Murayama, Phys. Rev. B **49**, 9009 (1994).
- [16] K. G. Singh and D. S. Rokhsar, Phys. Rev. B **46**, 3002 (1992).
- [17] R. Pai, R. Pandit, H. R. Krishnamurthy, and S. Ramasesha, Phys. Rev. Lett. **76**, 2937 (1996).
- [18] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, part 2*, Vol. 9 of *Landau course*, 3rd ed. (Pergamon Press, Oxford, 1980).
- [19] L. V. Butov *et al.*, Phys. Rev. Lett. **73**, 304 (1994).
- [20] E. I. Rashba and G. E. Gurgenishvili, Sov. Phys. Solid State **4**, 759 (1962).
- [21] J. Feldmann *et al.*, Phys. Rev. Lett. **59**, 2337 (1987).
- [22] D. S. Citrin, Phys. Rev. B **47**, 3832 (1993).
- [23] G. Björk, S. Pau, J. Jacobson, and Y. Yamamoto, Phys. Rev. B **50**, 17336 (1994).
- [24] B. L. V, in *The Physics of Semiconductors*, edited by R. Zimmermann (World Scientific, Singapore, 1996), p. 1927.
- [25] J. M. Ziman, *Models of Disorder* (Cambridge University Press, Cambridge, 1979).
- [26] S. Schmitt-Rink, D. S. Chemla, and D. A. B. Miller, Advances in Physics **38**, 89 (1989).
- [27] H. T. Stoof and M. Bijlsma, Phys. Rev. E **47**, 939 (1993).
- [28] W. Ketterle and N. J. van Druten, Phys. Rev. A **54**, 656 (1996).
- [29] N. E. Rehler and J. H. Eberly, Phys. Rev. A **3**, 1735 (1971).